Signals and Systems
Lecture 10

Outline

➢ **Computation of the convolution sum in time domain.**
- ✓ **Analytical Method.**
- ✓ **Array Method.**
- ✓ **Graphical Method.**
- ✓ **Using Matlab.**

Computation of the convolution sum in time domain

To compute the convolution sum of two signals, we can use, depending on the nature of the signals (causal or noncausal, finite duration or infinite, etc.), the following methods:

1. **Analytical Method.**
2. **Array Method.**
3. **Graphical Method.**
4. **Using Matlab.**

Analytical Method

If the signals are given by simple mathematical expression, we can analytically compute the convolution of signals.

Computation of actual convolution sums most commonly involves generalized exponential functions (including step functions and the sine/cosine functions). Two mathematical relationships that are frequently used are

a) $\sum_{n=0}^{\infty} \alpha^n = \dfrac{1}{1-\alpha}$, which converges only for $|\alpha| < 1$.

b) $\sum_{n=0}^{N-1} \alpha^n = \dfrac{1-\alpha^N}{1-\alpha}$, which is a finite sum and hence always converges (for all α).

Now let's consider the following pair of signals to be convolved:

$w[n] = a^n u[n]$ and $v[n] = b^n u[n]$, where $u[n]$ is the D-T unit step function, a and b are fixed nonzero real numbers.
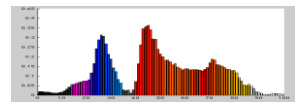
To solve this task, we must first find the closed-form expression of the convolved signals, $y[n] = w[n] * v[n] = \sum_{k=\infty}^{\infty} w[k] v[n-k]$

In this case, no matter which signal to fold and shift, so we fold the second signal $v[n]$.

Inserting $w[k] = a^k u[k]$ and $v[n-k] = b^{n-k} u[n-k]$ into the above equation, we get

$$y[n] = \sum_{k=-\infty}^{\infty} a^k u[k] b^{n-k} u[n-k], \quad n = 0,1,2,...$$

The nature of the signals $u[k]$ and $u[n-k]$ allow us to rewrite the following equation

$$y[n] = \sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k, \quad n = 0,1,2,\ldots$$

According to the values of a and b, we have two cases:

Case 1:

If $a = b \Rightarrow \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = n+1$ and the output $y[n] = b^n (n+1) = a^n (n+1), \quad n = 0,1,2,\ldots$

Substitute some values of n in the equation; we can calculate some value of the output

$$y[0] = a^0 (0+1)$$
$$y[1] = a^1 (1+1)$$
$$y[2] = a^2 (2+1)$$
$$.$$
$$.$$
$$.$$
$$y[n] = a^n (n+1)$$

Case 2:

If $a \neq b \Rightarrow \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = \dfrac{1 - \left(\dfrac{a}{b}\right)^{n+1}}{1 - \left(\dfrac{a}{b}\right)}$ and the output $y[n] = b^n \left(\dfrac{1 - \left(\dfrac{a}{b}\right)^{n+1}}{1 - \left(\dfrac{a}{b}\right)}\right), \quad n = 0,1,2,\ldots$

Examples of convolution sum

1) let's consider the following pair of functions to be convolved:

$x[n] = (0.8)^n u[n]$ and $h[n] = (0.3)^n u[n]$

**Solution:** to determine analytically the convolution $y[n] = x[n] * h[n]$ for

$x[n] = (0.8)^n u[n]$ and $h[n] = (0.3)^n u[n]$, we have: $y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m]$

Note that $x[m] = (0.8)^m u[m]$ and $h[n-m] = (0.3)^{n-m} u[n-m]$. Both $x[n]$ and $h[n]$ are causal. Therefore

$$y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] = \sum_{m=-\infty}^{\infty} (0.8)^m u[m] (0.3)^{n-m} u[n-m]$$

$$= \sum_{m=0}^{n} (0.8)^m (0.3)^{n-m}$$

In this summation, m lies between **0** and n $(0 \leq m \leq n)$. Therefore if $n \geq 0$ then both m and $n-m \geq 0$, so that $u[m] = u[n-m] = 1$. If $n < 0, m$ is negative because m lies between **0** and n, and $u[m] = 0$. Therefore

$$y[n] = \begin{cases} \sum_{m=0}^{n} (0.8)^m (0.3)^{n-m} & n \geq 0 \\ 0 & n < 0 \end{cases}$$
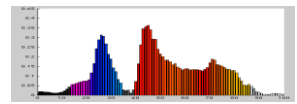
and $y[n] = (0.3)^n \sum_{m=0}^{n} \left(\frac{0.8}{0.3}\right)^m u[n]$. This is a geometric progression with common ratio $\left(\frac{0.8}{0.3}\right)$. We have

$$y[n] = (0.3)^n \frac{(0.8)^{n+1} - (0.3)^{n+1}}{(0.3)^n (0.8 - 0.3)} u[n] = 2\left[(0.8)^{n+1} - (0.3)^{n+1}\right] u[n]$$

2) Derived a closed-form expression for the convolution of $x[n]$ and $h[n]$ where

$$x[n] = \left(\frac{1}{6}\right)^{n-6} u[n]$$

and

$$h[n] = \left(\frac{1}{3}\right)^n u[n-3]$$

Solution: because both sequences are infinite in length, it is easier to evaluate the convolution sum directly:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

Note that because $x[n] = 0$ for $n < 0$ and $h[n] = 0$ for $n < 3$, $y[n]$ will be equal to zero for $n < 3$. Substituting $x[n]$ and $h[n]$ into the convolution sum, we have

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{6}\right)^{k-6} u[k] \left(\frac{1}{3}\right)^{n-k} u[n-k-3]$$

Due to the step $u[k]$, the lower limit on the sum may be changed to $k = 0$, and because $u[n-k-3]$ is zero for $k > n-3$, the upper limit may be changed to $k = n - 3$. Thus, for $n \geq 3$ the convolution sum becomes

$$y[n] = \sum_{k=0}^{n-3} \left(\frac{1}{6}\right)^{k-6} \left(\frac{1}{3}\right)^{n-k} = 6^6 \left(\frac{1}{3}\right)^n \sum_{k=0}^{n-3} \left(\frac{3}{6}\right)^k, \quad n \geq 3$$
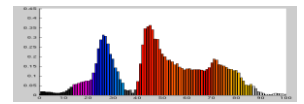
Using the geometric series to evaluate the sum, we have

$$y[n] = 6^6 \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{1}{2}\right)^{n-2}}{1 - \frac{1}{2}} = 2 \cdot 6^6 \left(\frac{1}{3}\right)^n \left[1 - 4\left(\frac{1}{2}\right)^n\right] \quad n \geq 3$$

3) Convolve $w[n] = u[n]$ and $h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$

To obtain the result analytically we must first decide which function to flip and shift. Normally we'd flip and shift the "simpler" time function, which in this case would be $w[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] w[n-k].$$

But in this case, we will flip and shift the more complicated function, just for illustration, causing the convolution sum to be written as $y[n] = \sum\limits_{k=-\infty}^{\infty} w[k]h[n-k]$

Given our definitions above for $w[n]$ and $h[n]$, we can write by substitution

$$h[n-k] = \left(\frac{1}{2}\right)^{n-k-3} u[n-k-3] \text{ and } w[k] = u[k]$$

Since $w[k] = 0$ for $k < 0$ and $h[n-k] = 0$ for $k > n-3$, we can see that the only region of "overlap" for which both $w[k]$ and $h[n-k]$ are nonzero is $0 \leq k \leq n-3$, Hence the convolution sum can be written as

$$y[n] = \sum_{k=-\infty}^{\infty} w[k]h[n-k] = \sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{n-k-3}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{\left(1 - 2^{n-2}\right)}{(1-2)}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3} \left(\left(\frac{1}{2}\right)^{2-n} - 1\right) = 2 - \left(\frac{1}{2}\right)^{n-3}$$

$$so, \quad y[n] = \left(2 - \left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$$

Array Method

This method is used frequently when both signals ($x[n]$ and $h[n]$) are finite and they are not necessary zero for all integers $n < 0$. Suppose

$$x[n] = 0, \quad n < p_1$$

$$h[n] = 0, \quad n < p_2$$

p_1 and p_2 are positive and/or negative numbers.
We can write the following equation:

$$x[n] * h[n] = \begin{cases} 0 & n < p_1 + p_2 \\ \sum\limits_{k=p_1}^{n-p_2} x[k]h[n-k] & n \geq p_1 + p_2 \end{cases}$$

This equation is still finite, so the convolution exists.
The convolution operation can be evaluated with an array as shown in figure 4-8. The elements of the array are filled in by multiplication column and row labels $\Rightarrow$ the values of $y[n]$ of the convolution determined by the sum of the elements in the backwards diagonals, where the diagonal that begins at $x[p_1 + i]$ and ends at $h[p_2 + i]$ is summed to yield the output $y[p_1 + p_2 + i]$.

*	$x[p_1]$	$x[p_1+1]$	$x[p_1+2]$	$x[p_1+3]$
$h[p_2]$	$x[p_1] \cdot h[p_2]$	$x[p_1+1] \cdot h[p_2]$	$x[p_1+2] \cdot h[p_2]$	$x[p_1+3] \cdot h[p_2]$
$h[p_2+1]$	$x[p_1] \cdot h[p_2+1]$	$x[p_1+1] \cdot h[p_2+1]$	$x[p_1+2] \cdot h[p_2+1]$	$x[p_1+3] \cdot h[p_2+1]$
$h[p_2+2]$	$x[p_1] \cdot h[p_2+2]$	$x[p_1+1] \cdot h[p_2+2]$	$x[p_1+2] \cdot h[p_2+2]$	$x[p_1+3] \cdot h[p_2+2]$
$h[p_2+3]$	$x[p_1] \cdot h[p_2+3]$	$x[p_1+1] \cdot h[p_2+3]$	$x[p_1+2] \cdot h[p_2+3]$	$x[p_1+3] \cdot h[p_2+3]$

$y[n]$

$y[p_1+p_2]$
$y[p_1+p_2+1]$
$y[p_1+p_2+2]$
$y[p_1+p_2+3]$
$y[p_1+p_2+4]$
$y[p_1+p_2+5]$
$y[p_1+p_2+6]$

Figure 4-8: Convolution Using Array Method
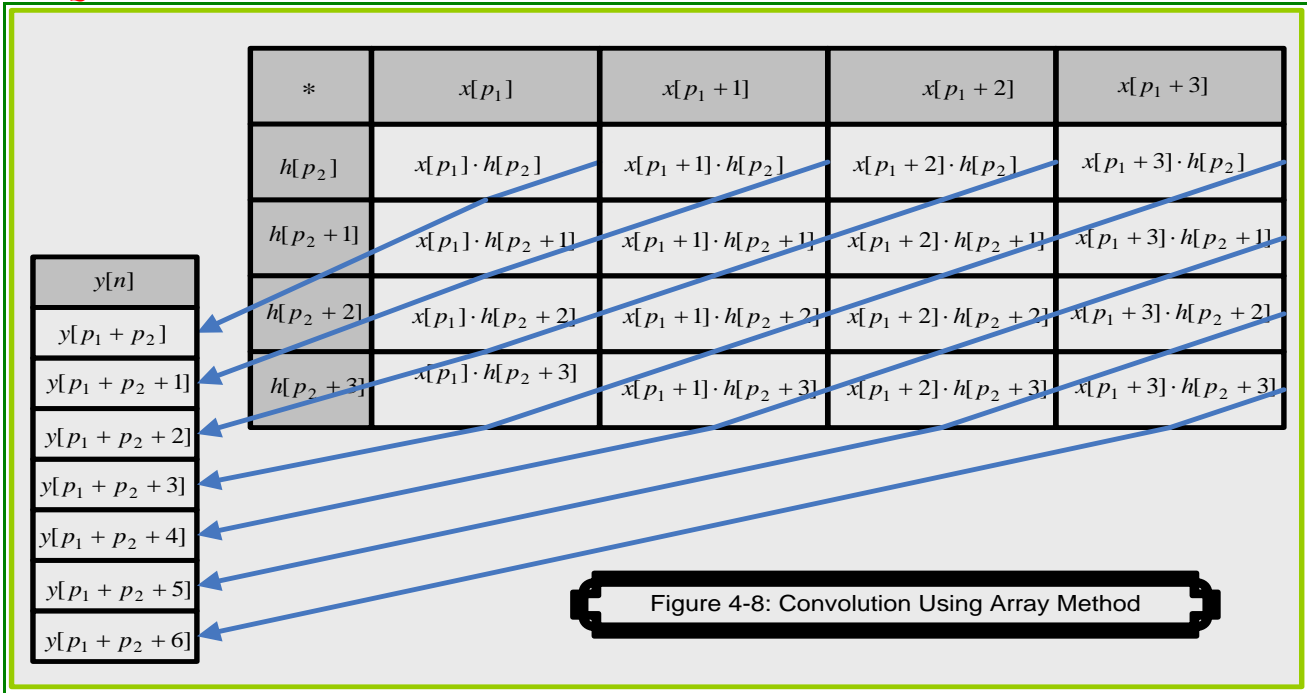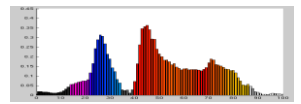
Example: to convolve the two sequences $x[n]$ and $h[n]$ depicted in figure 4-9 using **Array** method.
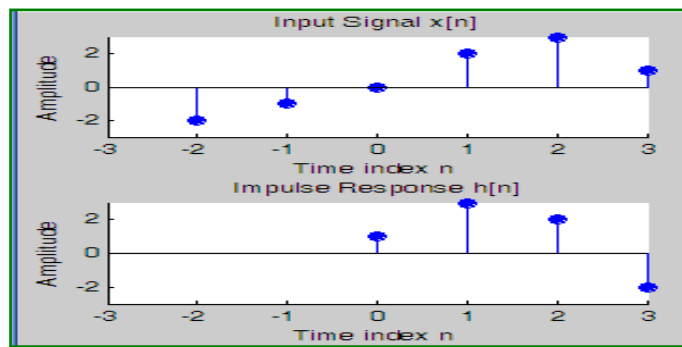

Figure 4-9

Solution:

In this case $p_1 = -2, \quad p_2 = 0 \Rightarrow$ the first element in $x[n]$: $n = -2$ and the first element in $h[n]$: $n = 0$, so the index for the $y[n]$ will be $-2 + 0 = -2$, then the first nonzero values of the output is the value $y[-2]$, The elements of the array are calculated as in **figure 4-10**.

		*	$x[-2]$ -2	$x[-1]$ -1	$x[0]$ 0	$x[1]$ 2	$x[2]$ 3	$x[3]$ 1
$y[-2]$	-2	$h[0]$ 1	-2	-1	0	2	3	1
$y[-1]$	-7	$h[1]$ 3	-6	-3	0	6	9	3
$y[0]$	-7	$h[2]$ 2	-4	-2	0	4	6	2
$y[1]$	4	$h[3]$ -2	4	2	0	-4	-6	-2
$y[2]$	11							
$y[3]$	14							
$y[4]$	5							
$y[5]$	-4							
$y[6]$	-2							

Figure 4-10: Convolution Using Array Method